Geometrothermodynamics of asymptotically Anti-de Sitter black holes

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ABSTRACT: We apply the formalism of geometrothermodynamics to the case of black holes with cosmological constant in four and higher dimensions. We use a thermodynamic metric which is invariant with respect to Legendre transformations and determines the geometry of the space of equilibrium states. For all known black holes in higher dimensions, we show that the curvature scalar of the thermodynamic metric in all the cases is proportional to the heat capacity. As a consequence, phase transitions, which correspond to divergencies of the heat capacity, are represented geometrically as true curvature singularities. We interpret this as a further indication that the curvature of the thermodynamic metric is a measure of thermodynamic interaction.

Contents

1. Introduction

One of the most interesting results of modern theoretical physics is its direct relation with many areas of mathematics like differential geometry. General relativity, for instance, can be considered mathematically as an application of differential geometry. Once a metric is given which is compatible with a torsion-free connection and satisfies Einstein's equations, the corresponding curvature turns out to be a measure of gravitational interaction. This is a fascinating result that combines apparently different concepts of geometry and physics, allowing us to study gravity by measuring the curvature of spacetime. In fact, this result can conceptually be generalized to include all the field interactions that are known in nature. The electromagnetic, weak, and strong interactions can classically be described by using the Minkowski metric and a gauge connection. In all the cases, the resulting gauge curvature can be considered as a measure of the corresponding field interaction (see, for instance, [1]).

During the last few decades several attempts have been made in order to introduce differential geometric concepts in ordinary thermodynamics. Hermann [2] formulated the concept of thermodynamic phase space as a differential manifold with a natural contact structure. In the thermodynamic phase space there exists a special subspace of thermodynamic equilibrium states. Weinhold [3] proposed an alternative approach in which in the space of equilibrium states a metric is introduced *ad hoc* as the Hessian of the internal energy. In an attempt to formulate the concept of thermodynamic length, Ruppeiner [4] introduced a metric which is conformally equivalent to Weinhold's metric. The study of the relation between the phase space and the metric structures of the space of equilibrium states led to the result that Weinhold's and Ruppeiner's thermodynamic metrics are not

invariant under Legendre transformations [5, 6], i.e. the geometric properties of the space of equilibrium states are different when different thermodynamic potentials are used. This result clearly contradicts ordinary equilibrium thermodynamics which is manifestly Legendre invariant. Moreover, the question whether the curvature of the space of equilibrium states can be considered as a measure for thermodynamic interaction remained unanswered. This was particularly clear in the case of the thermodynamics of black holes where a flat thermodynamic metric can be transformed into a non-flat metric by means of a Legendre transformation [7].

Recently, the formalism of geometrothermodynamics (GTD) was developed in order to unify in a consistent manner the geometric properties of the phase space and the space of equilibrium states [8]. Legendre invariance plays an important role in this formalism. In particular, it allows us to derive Legendre invariant metrics for the space of equilibrium states. It has been shown that there exist thermodynamic metrics which correctly describe the thermodynamic behavior of the ideal gas and the van der Waals gas. In fact, for the ideal gas the curvature vanishes whereas for the van der Waals gas the curvature is non-zero and diverges only at those points where phase transitions take place. Moreover, in the case of black hole thermodynamics in four dimensions we have shown recently [9] that there exists a thermodynamic metric with non-vanishing curvature which correctly describes the thermodynamic properties of those black holes. The main goal of the present work is to show that for all known asymptotically anti-de Sitter black holes in all dimensions there exists a thermodynamic metric with non-zero curvature which correctly describes the structure of phase transitions as dictated by the corresponding heat capacity. Consequently, our main result is that the curvature of the space of equilibrium states can be used in a general manner as a measure of the thermodynamic interaction of black holes.

The study of classical gravitational configurations on a background with cosmological constant has been intensified in the last few years. First, cosmological observations indicate that a positive cosmological constant could be responsible for the present acceleration of the Universe. On the other hand, a negative cosmological constant plays a distinguished role in the conjectured AdS/CFT correspondence, according to which 5-dimensional solutions of Einstein equations with negative cosmological constant can be used to derive certain statements about quantum field theory in four dimensions. In this context, the thermodynamic properties of black holes in an AdS background acquire especial importance since they give information about quantum field theory at non-zero temperature. Charged, rotating black holes in an AdS background are known explicitly only in four [10] and five dimensions [11]. Moreover, Reissner-Nordström-AdS and Kerr-AdS black holes are known in all dimensions [12, 13]. The thermodynamics of these higher dimensional black holes has been a subject of intensive investigation due to its importance in the context of the AdS/CFT conjecture [14–16]. In 2+1 gravity, the BTZ black hole presents an interesting phase transitions structure [17]. Charged topological AdS black holes and their phase transitions were analyzed in [18]. One of the interesting results is that all the intrinsic parameters that characterize black holes in higher dimensions can be treated, with certain care, as thermodynamic variables of ordinary thermodynamics.

This paper is organized as follows. In section 2 we review the general formalism of

GTD for black holes in arbitrary dimensions and introduce a Legendre invariant metric in the thermodynamic phase space which is used to generate the geometric structure of the space of equilibrium states. In sections 4 and 5 we investigate the structure of the phase transitions of the Reissner-Nordström AdS and Kerr AdS black holes, respectively, and show the points where phase transitions occur are characterized by curvature singularities of the thermodynamic metric. In the final section 7 we discuss our results. Throughout this paper we use units in which $G = c = k_B = \hbar = 1$.

2. Geometrothermodynamics

In order to describe a thermodynamic system with n degrees of freedom, we consider in GTD the thermodynamic phase space which is defined mathematically as a Riemannian contact manifold $(\mathcal{T}, \Theta, G)$, where $\mathcal{T}$ is a $(2n+1)$−dimensional manifold, Θ is a linear differential 1-form satisfying the condition $\Theta \wedge (d\Theta)^n \neq 0$, and G is a non-degenerate, Legendre invariant metric on $\mathcal{T}$. Here $\wedge$ represents the exterior product, d is the exterior derivative, and $(d\Theta)^n = d\Theta \wedge \ldots \wedge d\Theta$ (n−times). The submanifold $\mathcal{E} \subset \mathcal{T}$ defined by means of a smooth embedding mapping $\varphi : \mathcal{E} \longrightarrow \mathcal{T}$ such that the pullback $\varphi^*(\Theta) = 0$ is called the space of thermodynamic equilibrium states. A Riemannian structure g is induced naturally in $\mathcal{E}$ by means of $g = \varphi^*(G)$. It is then expected in GTD [8] that the physical properties of a thermodynamic system in a state of equilibrium can be described in terms of the geometric properties of the corresponding space of equilibrium states $\mathcal{E}$.

To be more specific we introduce in the phase space $\mathcal{T}$ the coordinates $Z^A = (\Phi, E^a, I^a)$ with $A = 0, \ldots, 2n$, and $a = 1, \ldots, n$. In ordinary thermodynamics, Φ corresponds to the thermodynamic potential, and E^a and I^a are the extensive and intensive variables, respectively. The fundamental differential form Θ can then be written in a canonical manner as $\Theta = d\Phi - \delta_{ab} I^a dE^b$, where δ_{ab} is the Euclidean metric. The metric components in $\mathcal{T}$ can be in general arbitrary C^2−functions of the coordinates, i. e., $G_{AB} = G_{AB}(Z^C)$. This arbitrariness is restricted by the condition that G must be invariant with respect to Legendre transformations. This is a necessary condition for our description of thermodynamic systems to be independent of the thermodynamic potential. This implies that $\mathcal{T}$ must be a curved manifold [8] because the special case of a metric with vanishing curvature turns out to be non Legendre invariant. Although in general any n−dimensional subset of the set of coordinates Z^A can be used to coordinatize the submanifold $\mathcal{E}$, for the sake of simplicity we choose the subset E^a as coordinates of $\mathcal{E}$. Then the smooth mapping $\varphi : \mathcal{E} \longrightarrow \mathcal{T}$ is given in terms of coordinates as $\varphi : \{E^a\} \longmapsto Z^A = \{\Phi(E^a), E^a, I^a(E^a)\}$. Consequently, the condition $\varphi^*(\Theta) = 0$ can be written as the expressions

$$d\Phi = \delta_{ab} I^a dE^b\,, \qquad \frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b\,, \tag{2.1}$$

which in ordinary thermodynamics correspond to the first law of thermodynamics and the conditions for thermodynamic equilibrium, respectively. We see that the specification of the mapping φ includes the specification of the relationship $\Phi = \Phi(E^a)$ that is nothing more but the fundamental equation from which all the information about a thermodynamic

system can be obtained [19]. The second law of thermodynamics is implemented in GTD as the convexity condition

$$\frac{\partial^2 \Phi}{\partial E^a \partial E^b} \geq 0 \ . \tag{2.2}$$

To complete our construction we need a metric G. There is a large arbitrariness in the selection of this metric since at this level it is only demanded that it satisfies the condition of Legendre invariance. For the sake of simplicity we will use the following choice

$$G = (d\Phi - \delta_{ab}I^a dE^b)^2 + (\delta_{ab}E^a I^b)(\eta_{cd}dE^c dI^d) , \qquad \eta_{ab} = \text{diag}(-1, 1, \ldots, 1) , \tag{2.3}$$

where η_{ab} is a pseudo-Euclidean metric in $\mathcal{E}$. This metric is a slight modification of the metric G^{II} presented in [8] which was used there to generate the simplest Legendre invariant generalizations of Weinhold's and Ruppeiner's thermodynamic metrics. It is easy to show that the metric (2.3) is invariant with respect to the Legendre transformation [20]

$$\{\Phi, E^a, I^a\} \longrightarrow \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\} \tag{2.4}$$

$$\Phi = \tilde{\Phi} - \delta_{ab}\tilde{E}^a \tilde{I}^b , \quad E^a = -\tilde{I}^a , \quad I^a = \tilde{E}^a , \tag{2.5}$$

and when "projected" on $\mathcal{E}$ by means of $g = \varphi^*(G)$ generates the thermodynamic metric

$$g = \left(E^c \frac{\partial \Phi}{\partial E^c}\right) \left(\eta_{ab}\delta^{bc}\frac{\partial^2 \Phi}{\partial E^c \partial E^d}dE^a dE^d\right) \ . \tag{2.6}$$

Once the fundamental equation $\Phi = \Phi(E^a)$ is known for a given thermodynamic system, the explicit form of the thermodynamic metric g can easily be computed. If the curvature of the thermodynamic metric is to be considered as a measure of the thermodynamic interaction, the metric (2.6) should be flat only for systems with no thermodynamic interaction. Moreover, phase transitions associated with divergencies of the thermodynamic interaction should correspond to curvature singularities. We will see that the metric (2.6) satisfies these conditions in the case of thermodynamic systems represented by black holes.

3. Four dimensional Kerr-Newman-AdS black hole

In the Einstein-Maxwell theory with cosmological constant Λ, which follows from the action

$$S_{EM} = \frac{1}{16\pi} \int_{M^4} d^4x [-\det(g_{\mu\nu})]^{1/2} (R - F_{\mu\nu}F^{\mu\nu} - 2\Lambda) , \tag{3.1}$$

the most general solution representing a black hole configuration is given by the Kerr-Newman-AdS solution [10] that in Boyer-Lindquist-like coordinates can be expressed as

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\varphi\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\varphi\right)^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) \tag{3.2}$$

where

$$\Delta_r = (r^2 + a^2)\left(1 + \frac{r^2}{l^2}\right) - 2mr + q^2 , \qquad \Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta , \tag{3.3}$$

$$\rho^2 = r^2 + a^2\cos^2\theta , \qquad \Xi = 1 - \frac{a^2}{l^2} \ . \tag{3.4}$$

This solution describes the gravitational field of a charged, rotating black hole with cosmological constant $\Lambda = -3/l^2$, where l is the curvature radius of the AdS spacetime. The electromagnetic potential A_μ is given as

$$A_t = -\frac{qr}{\rho^2}, \qquad A_\varphi = \frac{aqr\sin^2\theta}{\rho^2\Xi} \qquad (3.5)$$

with the angular, magnetic component A_φ that appears as a consequence of the rotation of the black hole. The horizons are determined by the (positive) roots of the equation $\Delta_r = 0$. In particular, the outer horizon is situated at $r = r_+$ and corresponds to the largest root.

The physical properties of this spacetime can be understood by considering the physical parameters entering the metric functions. The area of the horizon is a well-defined geometric parameter $A = \int \sqrt{g_{\varphi\varphi}g_{\theta\theta}}d\varphi d\theta$ that can easily be calculated at $r = r_+$ and yields $A = 4\pi(r_+^2 + a^2)/\Xi$. The surface gravity κ can be derived, modulo a trivial additive constant, from the equation $k^a k^b_{;a} - \kappa k^b = 0$ evaluated at the horizon for a timelike Killing vector field k^a. In the case under consideration we have that $k^a\partial_a = \partial_t + \Omega\partial_\varphi$, where Ω is the angular velocity measured by a non-rotating observer at infinity, so that the surface gravity is given as $\kappa = [3r_+^4 + (a^2 + l^2)r_+^2 - (a^2 + q^2)l^2]/[2l^2 r_+(r_+^2 + a^2)]$. The situation is more complicated in the case of the physical mass (total energy) M, angular momentum J, and electric charge Q, because these parameters are usually defined for asymptotically flat spacetimes. For asymptotically AdS spacetimes several definitions are possible and the issue has been clarified only recently by using the laws of black holes thermodynamics and the formalism of isolated horizons [15, 16]. It turns out that it is necessary to measure the angular velocity with respect to an observer which is not rotating at infinity [14]. Then the computation of the intrinsic physical parameters results in

$$M = \frac{m}{\Xi^2}, \qquad J = \frac{am}{\Xi^2}, \qquad Q = \frac{q}{\Xi} . \qquad (3.6)$$

The connection to thermodynamics arises when one considers the Bekenstein-Hawking entropy in terms of the horizon area, i.e. $S = A/4$. It is then easy to derive the generalized Smarr formula for the KN-AdS black hole

$$M^2 = J^2\left(\frac{1}{l^2} + \frac{\pi}{S}\right) + \frac{S^3}{4\pi^3}\left(\frac{1}{l^2} + \frac{\pi}{S} + \frac{\pi^2 Q^2}{S^2}\right)^2 , \qquad (3.7)$$

which is the fundamental thermodynamic equation. It relates the total energy M of the black hole with the extensive variables S, Q, and J. As in ordinary thermodynamics, in GTD it is the fundamental equation from which all the thermodynamic information can be derived. With the choice $E^a = \{S, Q, J\}$, the corresponding intensive variables become $I^a = \{T, \phi, \Omega\}$, where ϕ is the electric potential and Ω is the angular velocity. Furthermore, with this choice M corresponds to the thermodynamic potential. In this way, we have introduced all the coordinates $Z^A = \{M, S, Q, J, T, \phi, \Omega\}$ of the 7-dimensional thermodynamic phase space $\mathcal{T}$ which, according to eq. (2.3), becomes a Riemannian manifold with metric

$$G = (dM - TdS - \phi dQ - \Omega dJ)^2 + (ST + \phi Q + \Omega J)(-dSdT + dQd\phi + dJd\Omega) . \quad (3.8)$$

This is a non-degenerate metric with $\det(G_{AB}) = (ST + \Omega J + \phi Q)^4/16$ and non-zero curvature. Moreover, the contact structure of $\mathcal{T}$ is generated by the fundamental form $\Theta = dM - TdS - \phi dQ - \Omega dJ$. At the level of the phase space $\mathcal{T}$, the metric (3.8) plays an auxiliary role in the sense that it generates a Legendre invariant metric for the space of equilibrium states $\mathcal{E}$ with coordinates $\{E^a\}$. To this end, we introduce the smooth mapping

$$\varphi : \{S, Q, J\} \longmapsto \{M(S, Q, J), S, Q, J, T(S, Q, J), \phi(S, Q, J), \Omega(S, Q, J)\} \tag{3.9}$$

by using the fundamental equation (3.7) and the condition $\varphi^*(\Theta) = 0$ so that on $\mathcal{E}$ it corresponds to the first law of black hole thermodynamics $dM = TdS + \phi dQ + \Omega dJ$. This, in turn, can be used to compute the dual intensive variables corresponding to the temperature

$$T = \frac{\partial M}{\partial S} = \frac{S^2}{8M\pi^3}\left(\frac{1}{l^2} + \frac{\pi}{S} + \frac{\pi^2 Q^2}{S^2}\right)\left(\frac{3}{l^2} + \frac{\pi}{S} - \frac{\pi^2 Q^2}{S^2}\right) - \frac{\pi J^2}{2MS^2}\,, \tag{3.10}$$

the electric potential,

$$\phi = \frac{\partial M}{\partial Q} = \frac{QS}{2M\pi}\left(\frac{1}{l^2} + \frac{\pi}{S} + \frac{\pi^2 Q^2}{S^2}\right)\,, \tag{3.11}$$

and angular velocity

$$\Omega = \frac{\partial M}{\partial J} = \frac{J}{M}\left(\frac{1}{l^2} + \frac{\pi}{S}\right)\,. \tag{3.12}$$

Moreover, according to eq. (2.6), the metric structure of $\mathcal{E}$ is given as

$$g = (SM_S + QM_Q + JM_J)\begin{pmatrix} -M_{SS} & 0 & 0 \\ 0 & M_{QQ} & M_{QJ} \\ 0 & M_{QJ} & M_{JJ} \end{pmatrix}\,, \tag{3.13}$$

where subscripts represent partial derivative with respect to the corresponding coordinate. Notice that no cross terms of the form g_{SQ} or g_{SJ} appear in this expression, which would be proportional to M_{SQ} or M_{SJ}, respectively. This is due to the special choice of the auxiliary metric (3.8). Indeed, the minus sign in front of the term $dSdT$ in G leads to the disappearance of the cross terms of $g = \varphi^*(G)$ that involve the coordinate S, i. e., g_{SJ} and g_{SQ}. Our choice of G is in agreement with the condition of Legendre invariance and was inspired by inspecting the expression of the scalar curvature R. In fact, R always contains the determinant of the metric g in the denominator and, therefore, the zeros of $\det(g)$ could lead to curvature singularities (if those zeros are not canceled by the zeros of the numerator). On the other hand, as we will show below, the locations of the divergencies of the heat capacity coincide with the zeros of M_{SS}. Then, the choice of the metric (3.8) has the purpose of generating a metric g whose determinant is proportional to M_{SS}, leading to a one-to-one correspondence between the divergencies of the heat capacity and singularities of the scalar curvature.

In the thermodynamics of black holes, phase transitions must play an important role. Due to the absence of a realistic, microscopic model for the entropy of black holes, a problem

which is related to the absence of a theory of quantum gravity, there is no unanimity about the definition of phase transitions [21]. Nevertheless, one can adopt the point of view of ordinary thermodynamics and search for singular points in the behavior of thermodynamic variables. Such an approach was realized by Davies [22], showing that divergencies in the heat capacity indicate the points where phase transitions occur. We follow Davies' approach in this work. From the fundamental equation (3.7) it is straightforward to compute the heat capacity for the KN-AdS black hole:

$$C = T\frac{\partial S}{\partial T} = \frac{M_S}{M_{SS}} \tag{3.14}$$

$$= \frac{S\left(\frac{1}{l^2} + \frac{\pi}{S} + \frac{\pi^2 Q^2}{S^2}\right)\left(\frac{3}{l^2} + \frac{\pi}{S} - \frac{\pi^2 Q^2}{S^2}\right) - \frac{4\pi^4 J^2}{S^3}}{\left(\frac{1}{l^2} + \frac{\pi}{S} + \frac{\pi^2 Q^2}{S^2}\right)\left(\frac{6}{l^2} + \frac{\pi}{S}\right) - \left(\frac{\pi}{S} + \frac{2\pi^2 Q^2}{S^2}\right)\left(\frac{3}{l^2} + \frac{\pi}{S} - \frac{\pi^2 Q^2}{S^2}\right) + \frac{8\pi^3}{S^2}\left(\frac{\pi J^2}{S^2} - ST^2\right)}$$

Phase transitions are then determined by the roots of the denominator of C, i.e. $M_{SS} = 0$.

On the other hand, for the curvature of the metric (3.13) to be a measure of the thermodynamic interaction in the KN-AdS black hole, it must reproduce the phase transitions structure dictated by the heat capacity (3.14). To verify this property in an invariant manner we compute the scalar curvature of the thermodynamic metric (3.13), and notice that its denominator is given by

$$D_R = 4(SM_S + QM_Q + JM_J)^3(M_{QJ}^2 - M_{QQ}M_{JJ})^3 M_{SS}^2\,, \tag{3.15}$$

whereas the numerator is a rather cumbersome expression that can not be written in a compact form. We see that the denominator is proportional to the determinant of the metric (3.13). At first sight, the singular points of the heat capacity that are situated at $M_{SS} = 0$ correspond to true curvature singularities where the volume element vanishes. However, this is valid only if the numerator of the scalar curvature does not eliminate the zeros of the denominator. A numerical analysis shows that in fact the singularities of the heat capacity coincide with the singularities of the scalar curvature. We first fix the value of the cosmological constant and the entropy in such a way that we get for the charge and angular momentum an interval where the heat capacity diverges. The same process is then repeated for different combinations of values for the cosmological constant and entropy. As a result we find all the intervals where divergences occur. Around the divergent points of the heat capacity we then investigate the behavior of the thermodynamic scalar curvature and confirm that it becomes singular. We also noticed that the singular points coincide with the zeros of M_{SS} so that in fact the curvature singularities are situated at the points where phase transitions take place. The characteristic behavior of the heat capacity and curvature is depicted in figures 1 and 2.

In contrast to the above analysis, where the cosmological constant Λ has been treated as a fixed background parameter, it is possible to raise Λ at the level of an intrinsic parameter of the black hole and to consider it as an extensive thermodynamic variable [14]. This is an interesting possibility that follows from the Kaluza-Klein reduction of certain supergravity theories which are relevant in M-theory [23]. In GTD this possibility can easily be handled. In fact, in this case the phase space is 9-dimensional with coordinates $Z^A =$
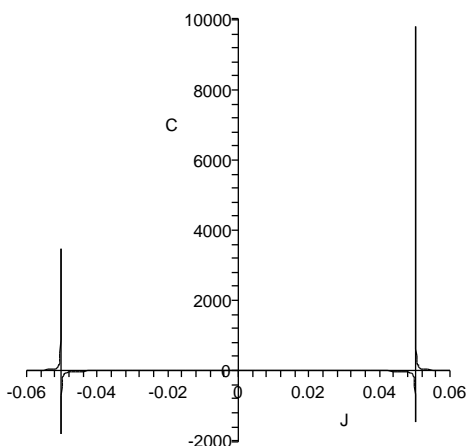
Figure 1: Characteristic behavior of the heat capacity C in terms of the angular momentum J. The chosen values are $\Lambda = -1$, $S = 1$, and $Q = 0.01$. The divergencies indicate points of phase transitions.
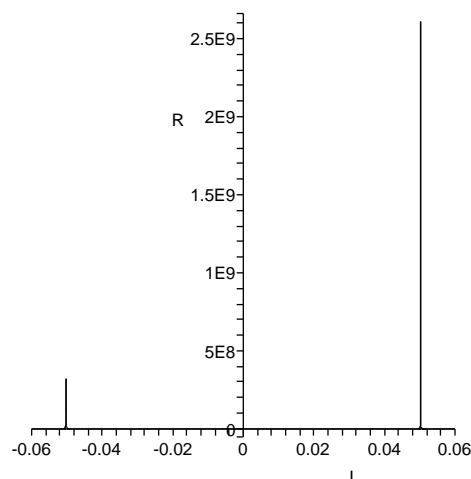
Figure 2: The thermodynamic scalar curvature R in terms of the angular momentum J. The values of the remaining parameters are as in figure 1. The singularity is located at the point of phase transition.

$\{M, S, Q, J, \Lambda, T, \phi, \Omega, L\}$, where L is the intensive coordinate dual to Λ. The construction of the Riemannian structure in $\mathcal{T}$ is straightforward, according to eq. (2.3). Furthermore, the coordinates of the space of equilibrium states can be chosen as $E^a = \{S, Q, J, \Lambda\}$ so that the corresponding thermodynamic metric becomes

$$g = (SM_S + QM_Q + JM_J + \Lambda M_\Lambda) \begin{pmatrix} -M_{SS} & 0 & 0 & 0 \\ 0 & M_{QQ} & M_{QJ} & M_{Q\Lambda} \\ 0 & M_{QJ} & M_{JJ} & M_{J\Lambda} \\ 0 & M_{Q\Lambda} & M_{J\Lambda} & M_{\Lambda\Lambda} \end{pmatrix} . \qquad (3.16)$$

The scalar curvature of this metric is again singular at the roots of $M_{SS} = 0$ so that, in principle, it can reproduce the structure of the phase transitions of the KN-AdS black hole.

4. Reissner-Nordström-AdS black hole in arbitrary dimensions

For a spacetime with arbitrary dimension D, the Einstein-Maxwell action with cosmological constant can be written as

$$S_{EM} = \frac{1}{16\pi} \int_{M^D} d^D x [-\det(g_{\mu\nu})]^{1/2} \left[R - F_{\mu\nu}F^{\mu\nu} + \frac{(D-1)(D-2)}{l^2} \right] , \qquad (4.1)$$

where l is the characteristic length of the AdS background that determines the cosmological constant by

$$\Lambda = -\frac{(D-1)(D-2)}{2l^2} . \qquad (4.2)$$

Then the metric for the RN-AdS black hole may be written in static coordinates as [24, 25]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{D-2}^2 \,, \tag{4.3}$$

where $d\Omega_{D-2}^2$ is the metric on the unit $(D-2)-$sphere. The function $f(r)$ can be expressed as

$$f(r) = 1 - \frac{\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}} + \frac{r^2}{l^2} \,, \tag{4.4}$$

where μ and q are intrinsic parameters related to the mass and charge of the black hole. This is an exact solution of Einstein-Maxwell equations with electromagnetic potential

$$A_t = - \left[\frac{D-2}{2(D-3)} \right]^{1/2} \frac{q}{r^{D-3}} \,. \tag{4.5}$$

The outer horizon is situated at $r = r_+$ where r_+ is the largest root of the equation $f(r) = 0$. From this algebraic equation it follows that

$$\mu = r_+^{D-3} + \frac{q^2}{r_+^{D-3}} + \frac{r_+^{D-1}}{l^2} \,. \tag{4.6}$$

Furthermore, the horizon area is given by

$$A = \omega_{D-2} r_+^{D-2} \,, \tag{4.7}$$

where $\omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma[(D-1)/2]$ is the volume of the unit $(D-2)-$sphere. The calculation of the physical mass and charge can be carried out either by using an appropriate generalization of the Arnowitt-Deser-Misner (ADM), which includes the case of asymptotically anti-de Sitter spacetimes [26, 27], or by using as a guide the laws of black hole thermodynamics [15]. The resulting quantities can be written as

$$M = \frac{(D-2)\omega_{D-2}}{16\pi} \mu \qquad Q = \frac{[2(D-2)(D-3)]^{1/2}\omega_{D-2}}{8\pi} q \,. \tag{4.8}$$

After some algebraic manipulations which involve the expressions given above for horizon area in the form $S = A/4$, the mass, charge, and the parameter μ, we obtain

$$M = \frac{(D-2)\omega_{D-2}}{16\pi} \left(\frac{4S}{\omega_{D-2}} \right)^{\frac{D-1}{D-2}} \left[\frac{1}{l^2} + \left(\frac{\omega_{D-2}}{4S} \right)^{\frac{2}{D-2}} + \frac{2\pi^2 Q^2}{(D-2)(D-3)S^2} \right] \,. \tag{4.9}$$

This is the fundamental equation for the RN-AdS black hole in arbitrary dimensions. In the special case $D = 4$ we recover the expression obtained in the last section. From the fundamental equation it is easy to derive all important thermodynamic variables. So we obtain the temperature

$$T = \frac{\partial M}{\partial S} = \frac{1}{4\pi} \left(\frac{4S}{\omega_{D-2}} \right)^{\frac{1}{D-2}} \left[\frac{(D-1)}{l^2} + (D-3) \left(\frac{\omega_{D-2}}{4S} \right)^{\frac{2}{D-2}} - \frac{2\pi^2 Q^2}{(D-2)S^2} \right] \,, \tag{4.10}$$

the electric potential,

$$\phi = \frac{\partial M}{\partial Q} = \frac{\pi Q}{(D-3)S} \left(\frac{4S}{\omega_{D-2}} \right)^{\frac{1}{D-2}} \,, \tag{4.11}$$

and the heat capacity

$$C = T\frac{\partial S}{\partial T} = \frac{M_S}{M_{SS}} = \frac{(D-2)S\left[\frac{D-1}{l^2} + (D-3)\left(\frac{\omega_{D-2}}{4S}\right)^{\frac{2}{D-2}} - \frac{2\pi^2 Q^2}{(D-2)S^2}\right]}{\frac{D-1}{l^2} - (D-3)\left(\frac{\omega_{D-2}}{4S}\right)^{\frac{2}{D-2}} + \frac{2(2D-5)\pi^2 Q^2}{(D-2)S^2}} \ . \tag{4.12}$$

These thermodynamic variables are then considered as independent quantities at the level of the 5-dimensional thermodynamic phase space $\mathcal{T}$ which can be coordinatized by $Z^A = \{M, S, Q, T, \phi\}$. The Riemannian structure of $\mathcal{T}$ is determined in this case by the metric

$$G = (dM - TdS - \phi dQ)^2 + (TS + \phi Q)\left(-dTdS + d\phi dQ\right) \ . \tag{4.13}$$

The space of equilibrium states $\mathcal{E}$ can be chosen as being determined by the simple mapping $\varphi : \{S, Q\} \mapsto \{M(S,Q), S, Q, T(S,Q), \phi(S,Q)\}$, where the explicit dependence of the intensive variables is as given above. The thermodynamic metric on $\mathcal{E}$ can be computed by means of the pullback $g = \varphi^*(G)$ that yields

$$g = (SM_S + QM_Q)\begin{pmatrix} -M_{SS} & 0 \\ 0 & M_{QQ} \end{pmatrix} \ . \tag{4.14}$$

As for the corresponding scalar curvature, we have performed an exhaustive analysis of the singularities and obtained a behavior similar to that of the heat capacity. Figures 3 and 4 show a characteristic example of the singular behavior of the heat capacity and the thermodynamic curvature. Our analysis shows that the curvature singularities reproduce the structure of the phase transitions of the Reissner-Nordström black hole in arbitrary dimensions.

5. Kerr-AdS black hole in arbitrary dimensions

In an arbitrary spacetime of dimension D, the Kerr-anti de Sitter metric is an exact solution to the equations $R_{\mu\nu} + (D-1)l^{-2}g_{\mu\nu} = 0$, which in Boyer-Lindquist coordinates can be expressed as [13]

$$ds^2 = -W\left(1 + \frac{r^2}{l^2}\right)dt^2 + \frac{2m}{U}\left(Wdt - \sum_{i=1}^{N}\frac{a_i\mu_i^2}{\Xi_i}d\phi_i\right)^2 + \sum_{i=1}^{N}\frac{r^2 + a_i^2}{\Xi_i}\mu_i^2 d\phi_i^2 \tag{5.1}$$

$$+ \frac{U}{V - 2m}dr^2 + \sum_{i=1}^{N+\epsilon}\frac{r^2 + a_i^2}{\Xi_i}d\mu_i^2 - \frac{1}{l^2W}\left(1 + \frac{r^2}{l^2}\right)^{-1}\left(\sum_{i=1}^{N+\epsilon}\frac{r^2 + a_i^2}{\Xi_i}\mu_i d\mu_i\right)^2 \ ,$$

where $N = [(D-1)/2]$ is the maximum number of independent rotational parameters a_i in N independent, orthogonal 2-planes, m is a parameter related to the total mass of the black hole, the parameter ϵ is defined as $\epsilon = (D-1) \bmod 2$, t is the time coordinate, r is the radial coordinate, ϕ_i are N different azimuthal angles, and μ_i are $(N + \epsilon)$ direction
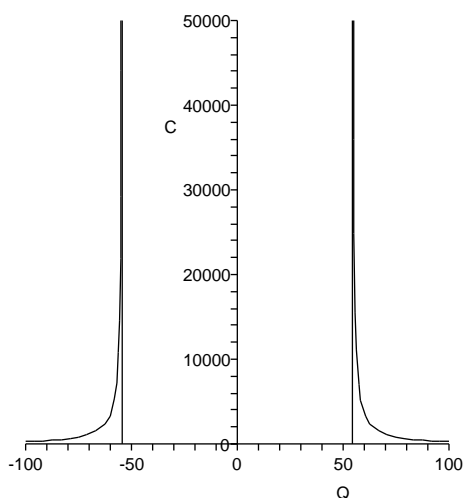
Figure 3: The heat capacity C as a function of the charge Q for the 10-dimensional RN-AdS black hole. Here $\Lambda = -1$ and $S = 100$. Phase transitions are clearly seen as divergencies of C.
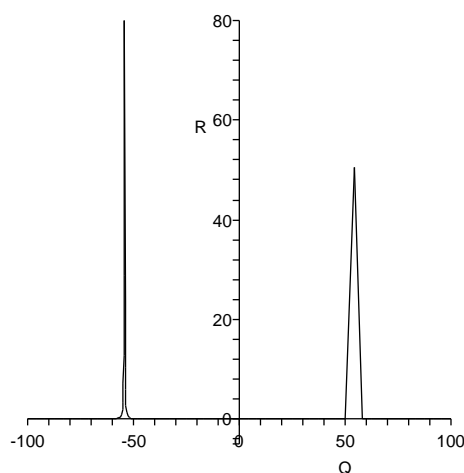

Figure 4: The scalar curvature R for the thermodynamic metric (4.14) as a function of the charge Q. The values of the remaining parameters are as in figure 3. The locations of curvature singularities and phase transitions coincide.

cosines. Moreover, the functions appearing in the metric are

$$W = \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{\Xi_i},$$

$$U = r^\epsilon \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{N}(r^2 + a_j^2), \qquad (5.2)$$

$$V = r^{\epsilon-2}\left(1 + \frac{r^2}{l^2}\right)\prod_{i=1}^{N}(r^2 + a_i^2),$$

$$\Xi_i = 1 - \frac{a_i^2}{l^2}. \qquad (5.3)$$

The outer horizon is situated at the radial distance $r = r_+$, where r_+ is the largest positive root of the algebraic equation

$$r_+^{\epsilon-2}\left(1 + \frac{r_+^2}{l^2}\right)\prod_{i=1}^{N}(r_+^2 + a_i^2) - 2m = 0, \qquad (5.4)$$

which once solved can be used to find explicitly the value of the entropy $S = A/4$. As in the previous cases, the presence of the cosmological constant requires especial care for the determination of the physical parameters. Let us denote by M the physical mass of the black hole and by J_i the momentum corresponding to the angular velocity Ω_i, measured by a non-rotating observer at infinity. Using as a guide the laws of thermodynamics, it can

be shown that [15]

$$M = \frac{m\omega_{D-2}}{4\pi} \left(\prod_{i=1}^{N} \Xi_i\right)^{-1} \left(\sum_{j=1}^{N} \frac{1}{\Xi_j} - \frac{1}{2}\right) , \qquad S = \frac{\omega_{D-2}}{4r_+} \prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i} , \qquad \text{for odd } D ,$$

(5.5)

$$M = \frac{m\omega_{D-2}}{4\pi} \left(\prod_{i=1}^{N} \Xi_i\right)^{-1} \sum_{j=1}^{N} \frac{1}{\Xi_j} , \qquad S = \frac{\omega_{D-2}}{4} \prod_{i=1}^{N} \frac{r_+^2 + a_i^2}{\Xi_i} , \qquad \text{for even } D ,$$

(5.6)

$$J_i = \frac{m a_i \omega_{D-2}}{4\pi \Xi_i} \left(\prod_{j=1}^{N} \Xi_j\right)^{-1} , \qquad \text{for arbitrary } D .$$

(5.7)

The next step is the derivation of a Smarr-like formula which relates all the physical parameters entering the metric, i.e. $M = M(S, J_i)$ or equivalently $S = S(M, J_i)$. To this end it is necessary to express the outer radius r_+ in terms of the physical parameters. In case this is possible, we first select a representation for the construction of GTD. Let us consider the $M-$representation. Then, the extensive thermodynamic variables are S and J_i. If we denote by T and Ω_i the corresponding dual intensive thermodynamic variables, the coordinates of the phase space $\mathcal{T}$ can be chosen as $Z^A = \{M, S, J_i, T, \Omega_i\}$ so that $\dim(\mathcal{T}) = 3 + 2[(D-1)/2]$, when the maximum number of angular momenta is considered. The Riemannian structure of $\mathcal{T}$ is completed with the metric

$$G = \left(dM - TdS - \sum_{i=1}^{N} \Omega_i dJ_i\right)^2 + \left(ST + \sum_{i=1}^{N} \Omega_i J_i\right) \left(-dSdT + \sum_{i=1}^{N} d\Omega_i dJ_i\right) . \quad (5.8)$$

The Riemannian submanifold of equilibrium states $\mathcal{E}$ is defined by means of the smooth mapping $\varphi : \{S, J_i\} \longmapsto \{M(S, J_i), S, J_i, T(S, J_i), \Omega_i(S, J_i)\}$, which induces the first law of thermodynamics $dM = TdS + \sum_{i=1}^{N} \Omega_i dJ_i$ and the $(N+1)-$dimensional metric

$$g = \left(SM_S + \sum_{i=1}^{N} J_i M_{J_i}\right) \begin{pmatrix} -M_{SS} & 0 & 0 & 0 \\ 0 & M_{J_1 J_1} & \dots & M_{J_1 J_N} \\ 0 & . & \dots & . \\ 0 & M_{J_N J_1} & \dots & M_{J_N J_N} \end{pmatrix} .$$

(5.9)

It is straightforward to see that the curvature scalar of this thermodynamic metric contains the term

$$\left(SM_S + \sum_{i=1}^{N} J_i M_{J_i}\right)^N [\det(M_{ij})]^N M_{SS}^2$$

(5.10)

in its denominator, where $M_{ij} = \partial^2 M/(\partial J_i \partial J_j)$, so that curvature singularities could appear at the points where the condition $M_{SS} = 0$ is satisfied. On the other hand, as mentioned above, in the mass representation the heat capacity can be expressed as $C = M_S/M_{SS}$. It follows then that phase transitions that occur when the heat capacity diverges, could be represented as curvature singularities of the thermodynamic metric. This holds, of

course, only if the expressions appearing in the numerator of the curvature do not eliminate the zeros of M_{SS}. We note that in (5.10) the term in round brackets cannot be zero because it can be shown to be proportional to the total mass M as a result of Euler's identity. As for the determinant of M_{ij}, its zeros, if any, can be found only if the fundamental equation $M = M(S, J_i)$ can be written explicitly.

The determination of the fundamental equation turns out to be a non-trivial problem. In fact, one way to determine it is to compute the solutions of the algebraic equation (5.4) which, in general, is a polynomial of order D in r_+. We were not able to find explicit solutions. However, the case of vanishing cosmological constant ($l^2 \to \infty$) and only one non-vanishing rotational parameter, say, $a_1 = a$, can be manipulated explicitly. From eqs. (5.4)– (5.7), we obtain for this specific case

$$(r_+^2 + a^2)r_+^{D-5} - 2m = 0 \,, \tag{5.11}$$

$$M = \frac{m\omega_{D-2}}{4\pi}(D/2 - 1) \,,$$

$$S = \frac{\omega_{D-2}}{4}(r_+^2 + a^2)r_+^{D-4} \,,$$

$$J = \frac{ma\omega_{D-2}}{4\pi} \,. \tag{5.12}$$

A straightforward manipulation of these equations results in the expression

$$M = \frac{D/2 - 1}{\pi} \left(\frac{\omega_{D-2}}{2^D}\right)^{\frac{1}{D-2}} S^{\frac{D-3}{D-2}} \left(1 + 4\pi^2 \frac{J^2}{S^2}\right)^{\frac{1}{D-2}} \,, \tag{5.13}$$

which constitutes the corresponding fundamental equation. In turn, the heat capacity can be expressed as

$$C = -\frac{(D-2)S \left[3S^2 + 20\pi^2 J^2 - D(S^2 + 4\pi^2 J^2)\right] \left(S^2 + 4\pi^2 J^2\right)}{3S^4 + 24\pi^2 J^2 S^2 + 240\pi^4 J^4 - D(S^4 + 48\pi^4 J^4)} \,. \tag{5.14}$$

The corresponding thermodynamic metric of the space of equilibrium states reduces to the 2-dimensional metric

$$g = (SM_S + JM_J)\left(-M_{SS}dS^2 + M_{JJ}dJ^2\right) \,, \tag{5.15}$$

independently of the dimension D. The expression for the scalar curvature cannot be transformed into a compact form. Only the denominator can be shown to contain the expression

$$(D-3)[3S^4 + 24\pi^2 J^2 S^2 + 240\pi^4 J^4 - D(S^4 + 48\pi^4 J^4)]^2 \,, \tag{5.16}$$

which determines the phase transition structure of the heat capacity (5.14). To see that the singular behavior of the scalar curvature coincides with that of the heat capacity we perform a detailed numerical analysis of both expressions. The characteristic singular behavior is depicted in figures 5 and 6. For all analyzed regions a similar behavior was detected, showing that in fact the points where phase transitions occur are characterized by curvature singularities of the thermodynamic metric.
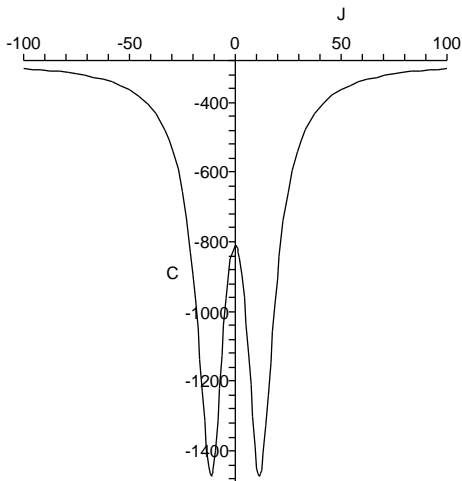
Figure 5: The heat capacity C as a function of the angular momentum J for the 10-dimensional Kerr black with one rotational parameter. Here $\Lambda = 0$ and $S = 100$.
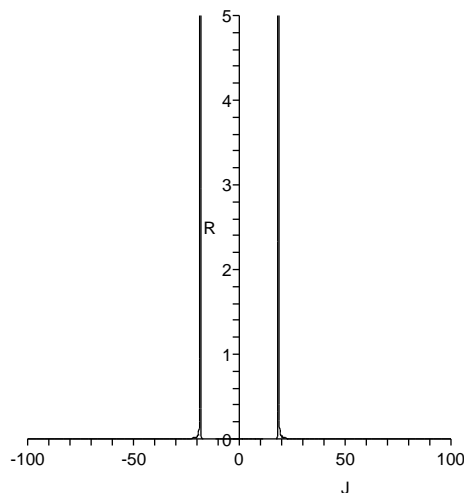

Figure 6: The scalar curvature R for the thermodynamic metric (5.15) as a function of the angular momentum J. The remaining parameters are chosen as in figure 5.

6. The role of statistical ensembles

In the above applications of GTD of black holes, the starting point of the analysis is the fundamental thermodynamic equation. Nevertheless, it is known that the thermodynamic properties of black holes can drastically depend on the choice of statistical ensemble [14, 24, 25, 32]. The fact that different ensembles lead to different heat capacities implies that the phase transitions structure depends on the statistical model under consideration. The question arises whether GTD is able to correctly handle this dependence. We will show in this section that the answer is in the affirmative. For the sake of concreteness and simplicity we will show this in the explicit example of the RN-AdS black hole in $D = 4$.

Let us first consider the canonical ensemble which is characterized by a fixed charge. In this case, according to eqs. (4.9)–(4.11), the relevant thermodynamic variables can be expressed as

$$M = \frac{r_+}{2}\left(1 + \frac{Q^2}{r_+^2} + \frac{r_+^2}{l^2}\right) , \quad T = \frac{1}{4\pi}\left(\frac{1}{r_+} - \frac{Q^2}{r_+^3} + \frac{3r_+^2}{l^2}\right) , \quad \phi = \frac{Q}{r_+} . \qquad (6.1)$$

where we have used the relation $S = \pi r_+^2$. Then, the heat capacity $C_Q = (\partial M/\partial T)_Q$ becomes

$$C_Q = 2\pi r_+^2 \left[\frac{3r_+^4 + l^2(r_+^2 - Q^2)}{3r_+^4 - l^2(r_+^2 - 3Q^2)}\right] . \qquad (6.2)$$

The analysis of this case in the context of GTD can be carried out as in section 4. The metric G of the phase space $\mathcal{T}$ is given by (4.13), whereas the metric g of the space of

equilibrium states $\mathcal{E}$ coincides with (4.14) and can be written explicitly as

$$
\begin{aligned}
g &= (SM_S + QM_Q)\left(-M_{SS}dS^2 + M_{QQ}dQ^2\right) \\
&= \frac{3r_+^4 + l^2(3Q^2 + r_+^2)}{4l^2r_+^2}\left[-\frac{3r_+^4 + l^2(3Q^2 - r_+^2)}{8\pi^2l^2r_+^4}\,dS^2 + dQ^2\right] \, .
\end{aligned}
\tag{6.3}
$$

This metric determines the geometric properties of the manifold $\mathcal{E}$. In particular, the curvature scalar can be expressed in the compact form

$$
R = -\frac{48l^4r_+^4\mathcal{N}(r_+, Q)}{[3r_+^4 + l^2(r_+^2 + 3Q^2)]^3[3r_+^4 - l^2(r_+^2 - 3Q^2)]^2}\, ,
\tag{6.4}
$$

where

$$
\mathcal{N}(r_+, Q) = 3l^4Q^4(2l^2 - 21r_+^2) - 3l^2r_+^2Q^2(2l^4 - 5l^2r_+^2 + 6r_+^4) - r_+^6(2l^4 + 3l^2r_+^2 - 45r_+^4).
\tag{6.5}
$$

The curvature singularities are then given by the zeros of the equation

$$
3r_+^4 - l^2(r_+^2 - 3Q^2) = 0
\tag{6.6}
$$

which are also the blow-up points of the heat capacity (6.2). This shows that in the canonical ensemble approach the structure of the phase transitions coincides with the singular structure of the thermodynamic curvature.

We now consider the grand canonical ensemble that corresponds to a fixed electric potential ϕ. To obtain the grand canonical potential one can use the Euclidean action method as in [25] or, equivalently, apply a Legendre transformation as in [32] which interchanges the role of the variables Q and ϕ. That is to say, one introduces a new thermodynamic potential $\tilde{M}$ by means of the Legendre transformation

$$
\tilde{M} = M - \phi Q \, .
\tag{6.7}
$$

In [32], the potential $\tilde{M}$ is interpreted as the internal energy. Then, from eq. (6.1) we obtain

$$
\tilde{M} = \frac{r_+}{2}\left(1 - \phi^2 + \frac{r_+^2}{l^2}\right) \, .
\tag{6.8}
$$

On the other hand, calculating the total differential of $\tilde{M}$ and using the first law of thermodynamics, we obtain $d\tilde{M} = TdS - Qd\phi$ so that the dual thermodynamic variables $T = \partial\tilde{M}/\partial S$ and $Q = -\partial\tilde{M}/\partial\phi$ can be written as

$$
T = \frac{1}{4\pi r_+}\left(1 - \phi^2 + \frac{3r_+^2}{l^2}\right) \, , \quad Q = \phi r_+ \, .
\tag{6.9}
$$

Furthermore, it is straightforward to calculate the heat capacity

$$
C_\phi = \left(\frac{\partial\tilde{M}}{\partial T}\right)_\phi = 2\pi r_+^2\left[\frac{3r_+^2 + l^2(1 - \phi^2)}{3r_+^2 - l^2(1 - \phi^2)}\right] \, .
\tag{6.10}
$$

Hence, the blow-up points that determine the phase transitions structure coincide with the solutions of the equation

$$3r_+^2 - l^2(1 - \phi^2) = 0 \ . \tag{6.11}$$

We now analyze the case of the grand canonical ensemble in GTD. The new thermodynamic potential is $\tilde{M} = \tilde{M}(S, \phi)$ so that the coordinates in the phase space $\mathcal{T}$ are $Z^A = \{\tilde{M}, E^a, I^a\} = \{\tilde{M}, S, \phi, T, -Q\}$. Then, the fundamental contact form can be written it its canonical form as $\Theta = d\tilde{M} - TdS + Qd\phi$. The particular metric (2.3) we are using here for the geometry of the manifold $\mathcal{T}$ takes the form

$$G = \left(d\tilde{M} - TdS + Qd\phi\right)^2 - (ST - \phi Q)(dSdT + d\phi dQ) \ . \tag{6.12}$$

The submanifold $\mathcal{E}$ of equilibrium states with coordinates $E^a = \{S, \phi\}$ is defined by means of the smooth map $\varphi : \mathcal{E} \longrightarrow \mathcal{T}$ that in this case implies the explicit dependence

$$\varphi : \{E^a\} \longmapsto \{Z^A(E^a)\} = \left\{\tilde{M}(S, \phi), S, \phi, T = \frac{\partial \tilde{M}}{\partial S}, -Q = \frac{\partial \tilde{M}}{\partial \phi}\right\} \ . \tag{6.13}$$

In turn, the geometric properties of $\mathcal{E}$ are described by the metric $g = \varphi^*(G)$ which becomes

$$\begin{aligned} g &= \left(S\tilde{M}_S + \phi\tilde{M}_\phi\right)\left(-\tilde{M}_{SS}dS^2 + \tilde{M}_{\phi\phi}d\phi^2\right) \\ &= -\frac{3r_+^2 - l^2(5\phi^2 - 1)}{4l^2}\left[\frac{3r_+^2 - l^2(1 - \phi^2)}{8\pi^2 l^2 r_+^2}\, dS^2 + r_+^2\, d\phi^2\right] \end{aligned} \tag{6.14}$$

A straightforward calculation of the scalar curvature for this metric yields

$$R = -\frac{16l^4\mathcal{N}(r_+, \phi)}{r_+^2[3r_+^2 - l^2(5\phi^2 - 1)]^3[3r_+^2 - l^2(1 - \phi^2)]^2} \ , \tag{6.15}$$

where

$$\mathcal{N}(r_+, \phi) = 5l^4\phi^4(l^2 + 24r_+^2) + 3l^2 r_+^2\phi^2(60r_+^2 - 47l^2) + 3(l^6 + l^4 r_+^2 - 6l^2 r_+^4 + 36r_+^6) \ . \tag{6.16}$$

From the denominator of this expression and eq. (6.11), we conclude that there exist curvature singularities at those points where the heat capacity C_ϕ blows up. There is an additional term in the denominator which under the condition $3r_+^2 - l^2(5\phi^2 - 1) = 0$ can, in principle, lead to curvature singularities. However, it can easily be seen that this term is proportional to the conformal factor of the metric, i. e., $S\tilde{M}_S + \phi\tilde{M}_\phi$, which in turn, according to Euler's identity, is proportional to the internal energy $\tilde{M}$ of the black hole. This seems to prohibit the appearance of additional singularities.

We conclude that GTD can handle correctly the different thermodynamic schemes that follow from different statistical ensembles for the RN-AdS black hole in $D = 4$. Similar analysis can be performed for other types of black holes and we expect similar results. The fact that in general the denominator of the scalar curvature contains the denominator of the heat capacity is an indication that, even in the case of different statistical ensembles, there exists a correspondence between singular points of the heat capacity and curvature singularities.

7. Discussion and conclusions

The main result of this work is that in the space of equilibrium states of all known asymptotically anti-de Sitter black holes in arbitrary dimensions there exists a thermodynamic metric whose curvature is singular at those points where phase transitions of the heat capacity occur. This has been shown by considering a particular metric in the thermodynamic phase space, and applying the formalism of geometrothermodynamics. An important property of our choice of thermodynamic metric is that it is invariant with respect to Legendre transformations so that the properties of our geometric description of thermodynamics are independent of the choice of thermodynamic potential and representation.

The explicit examples considered in this work include the Reissner-Nordström black holes and the Kerr black holes on an anti-de Sitter background in arbitrary dimensions ($D \geq 4$). In the last example, we were able to explicitly analyze only the case of one rotational parameter with vanishing cosmological constant. The most general case of non-vanishing cosmological constant and arbitrary number of rotational parameters leads to a set of algebraic equations which we were not able to solve. Hence the fundamental thermodynamic equation cannot be written explicitly. Nevertheless, even in this general case our results show that if the numerator of the resulting expression for the curvature scalar does not cancel the zeros of the denominator, then the curvature singularities are situated at those points where the heat capacity diverges, a fact that in ordinary thermodynamics is considered as an indication of a phase transition. The four-dimensional Kerr-Newman black hole on an anti-de Sitter background was also analyzed, obtaining similar results. The Kerr-Newman-AdS black hole is not known in higher dimensions. Thus, the examples considered in this work include all known higher dimensional solutions for black holes on an anti-de Sitter background.

The starting point in our geometrothermodynamical analysis is the metric G of the thermodynamic phase space. In this work we used the particular metric (2.3) which was chosen such that the metric g of the space of equilibrium states takes the specific form (2.6), whose determinant becomes proportional to $\partial^2 \Phi / \partial E^1 \partial E^1$. In fact, the use of the metric η_{ab}, instead of δ_{ab}, in G guarantees that all non-diagonal terms of the form $\partial^2 \Phi / \partial E^1 \partial E^k$, with $k \neq 1$, vanish. Furthermore, it is known that the scalar curvature always contains terms with the determinant of the metric in their denominator. It can therefore be expected that there exist true curvature singularities at those points where $\partial^2 \Phi / \partial E^1 \partial E^1 = 0$. We use in this work the $M-$representation, in which $\Phi = M$, and choose $E^1 = S$ so that the singularities are expected at $\partial^2 M / \partial S^2 = M_{SS} = 0$. On the other hand, in this representation the heat capacity can be written as $C = M_S/M_{SS}$ with divergencies at $M_{SS} = 0$. For all known asymptotically anti-de Sitter black holes we have shown that the remaining terms of the thermodynamic scalar curvature do not cancel the zeros of M_{SS}. Thus we conclude that the curvature of the space of thermodynamic equilibrium states can be interpreted in an invariant manner as a measure of the thermodynamic interaction. This is in contrast with other studies [28, 29] where the curvature of the space of equilibrium states depends on the choice of a specific representation so that, for instance, a flat thermodynamic metric can be associated to a system with non-trivial thermodynamic interaction.

In the case of the RN-AdS black hole in $D = 4$, we analyzed the different thermodynamics which follow from the choice of different statistical ensembles. In particular, we considered the canonical ensemble, with fixed charge Q, and the grand canonical ensemble, with fixed electric potential ϕ. We obtained the corresponding heat capacities C_Q and C_ϕ, which lead to different phase transitions structures. According to our general procedure of GTD for black holes, we constructed explicitly for both ensembles the corresponding phase manifolds and the manifolds of equilibrium states. In both cases we obtained that the there exists a correspondence between singular points of the heat capacity and curvature singularities of the thermodynamic metric. We expect similar results in the analysis of more general black holes. The fact that in general the denominator of the scalar curvature contains the denominator of the heat capacity is an indication that GTD can handle correctly the different thermodynamic schemes that follow from different statistical ensembles.

Legendre invariance is an important element of our approach. It limits the number of metrics that can be used to describe ordinary thermodynamics in terms of geometric concepts. It is also essential in order to obtain results that are independent of the choice of extensive variables and thermodynamic potential. A different point of view, in which for a given thermodynamic system there exists a preferred thermodynamic potential [30, 31], is necessary in order to explain the vanishing of Ruppeiner's thermodynamic curvature in cases where thermodynamic interaction is present as, for instance, in Reissner-Nordström black holes. We believe that ordinary thermodynamics, which is Legendre invariant, must not be changed when one tries to represent it in terms of geometric concepts.

The computer algebra system REDUCE 3.8 was used for most of the calculations reported in this work.

Acknowledgments

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